Contents

Preface to the C++ Edition xi
Preface to the Second Edition xii
Preface to the First Edition xv
License Information xvii
Computer Programs by Chapter and Section xxi

1 Preliminaries 1
1.0 Introduction 1
1.1 Program Organization and Control Structures 5
1.2 Some C++ Conventions for Scientific Computing 16
1.3 Implementation of the Vector and Matrix Classes 25
1.4 Error, Accuracy, and Stability 31

2 Solution of Linear Algebraic Equations 35
2.0 Introduction 35
2.1 Gauss-Jordan Elimination 39
2.2 Gaussian Elimination with Backsubstitution 44
2.3 LU Decomposition and Its Applications 46
2.4 Tridiagonal and Band Diagonal Systems of Equations 53
2.5 Iterative Improvement of a Solution to Linear Equations 58
2.6 Singular Value Decomposition 62
2.7 Sparse Linear Systems 74
2.8 Vandermonde Matrices and Toeplitz Matrices 93
2.9 Cholesky Decomposition 99
2.10 QR Decomposition 101
2.11 Is Matrix Inversion an $N^3$ Process? 105

3 Interpolation and Extrapolation 108
3.0 Introduction 108
3.1 Polynomial Interpolation and Extrapolation 111
3.2 Rational Function Interpolation and Extrapolation 114
3.3 Cubic Spline Interpolation 116
3.4 How to Search an Ordered Table 120
3.5 Coefficients of the Interpolating Polynomial 123
3.6 Interpolation in Two or More Dimensions 126
4 Integration of Functions

4.0 Introduction
4.1 Classical Formulas for Equally Spaced Abscissas
4.2 Elementary Algorithms
4.3 Romberg Integration
4.4 Improper Integrals
4.5 Gaussian Quadratures and Orthogonal Polynomials
4.6 Multidimensional Integrals

5 Evaluation of Functions

5.0 Introduction
5.1 Series and Their Convergence
5.2 Evaluation of Continued Fractions
5.3 Polynomials and Rational Functions
5.4 Complex Arithmetic
5.5 Recurrence Relations and Clenshaw’s Recurrence Formula
5.6 Quadratic and Cubic Equations
5.7 Numerical Derivatives
5.8 Chebyshev Approximation
5.9 Derivatives or Integrals of a Chebyshev-approximated Function
5.10 Polynomial Approximation from Chebyshev Coefficients
5.11 Economization of Power Series
5.12 Padé Approximants
5.13 Rational Chebyshev Approximation
5.14 Evaluation of Functions by Path Integration

6 Special Functions

6.0 Introduction
6.1 Gamma Function, Beta Function, Factorials, Binomial Coefficients
6.2 Incomplete Gamma Function, Error Function, Chi-Square
6.3 Exponential Integrals
6.4 Incomplete Beta Function, Student’s Distribution, F-Distribution,
    Cumulative Binomial Distribution
6.5 Bessel Functions of Integer Order
6.6 Modified Bessel Functions of Integer Order
6.7 Bessel Functions of Fractional Order, Airy Functions, Spherical
    Bessel Functions
6.8 Spherical Harmonics
6.9 Fresnel Integrals, Cosine and Sine Integrals
6.10 Dawson’s Integral
6.11 Elliptic Integrals and Jacobian Elliptic Functions
6.12 Hypergeometric Functions

7 Random Numbers

7.0 Introduction
7.1 Uniform Deviates
7.2 Transformation Method: Exponential and Normal Deviates 291
7.3 Rejection Method: Gamma, Poisson, Binomial Deviates 294
7.4 Generation of Random Bits 300
7.5 Random Sequences Based on Data Encryption 304
7.6 Simple Monte Carlo Integration 308
7.7 Quasi- (that is, Sub-) Random Sequences 313
7.8 Adaptive and Recursive Monte Carlo Methods 320

8 Sorting 332
8.0 Introduction 332
8.1 Straight Insertion and Shell’s Method 333
8.2 Quicksort 336
8.3 Heapsort 339
8.4 Indexing and Ranking 341
8.5 Selecting the $M$th Largest 344
8.6 Determination of Equivalence Classes 348

9 Root Finding and Nonlinear Sets of Equations 351
9.0 Introduction 351
9.1 Bracketing and Bisection 354
9.2 Secant Method, False Position Method, and Ridders’ Method 358
9.3 Van Wijngaarden–Dekker–Brent Method 363
9.4 Newton-Raphson Method Using Derivative 366
9.5 Roots of Polynomials 373
9.6 Newton-Raphson Method for Nonlinear Systems of Equations 383
9.7 Globally Convergent Methods for Nonlinear Systems of Equations 387

10 Minimization or Maximization of Functions 398
10.0 Introduction 398
10.1 Golden Section Search in One Dimension 401
10.2 Parabolic Interpolation and Brent’s Method in One Dimension 406
10.3 One-Dimensional Search with First Derivatives 410
10.4 Downhill Simplex Method in Multidimensions 413
10.5 Direction Set (Powell’s) Methods in Multidimensions 417
10.6 Conjugate Gradient Methods in Multidimensions 424
10.7 Variable Metric Methods in Multidimensions 430
10.8 Linear Programming and the Simplex Method 434
10.9 Simulated Annealing Methods 448

11 Eigensystems 461
11.0 Introduction 461
11.1 Jacobi Transformations of a Symmetric Matrix 468
11.2 Reduction of a Symmetric Matrix to Tridiagonal Form: Givens and Householder Reductions 474
11.3 Eigenvalues and Eigenvectors of a Tridiagonal Matrix 481
11.4 Hermitian Matrices 486
11.5 Reduction of a General Matrix to Hessenberg Form 487
11.6 The QR Algorithm for Real Hessenberg Matrices 491
11.7 Improving Eigenvalues and/or Finding Eigenvectors by Inverse Iteration 498

12 Fast Fourier Transform 501
12.0 Introduction 501
12.1 Fourier Transform of Discretely Sampled Data 505
12.2 Fast Fourier Transform (FFT) 509
12.3 FFT of Real Functions, Sine and Cosine Transforms 515
12.4 FFT in Two or More Dimensions 526
12.5 Fourier Transforms of Real Data in Two and Three Dimensions 530
12.6 External Storage or Memory-Local FFTs 536

13 Fourier and Spectral Applications 542
13.0 Introduction 542
13.1 Convolution and Deconvolution Using the FFT 543
13.2 Correlation and Autocorrelation Using the FFT 550
13.3 Optimal (Wiener) Filtering with the FFT 552
13.4 Power Spectrum Estimation Using the FFT 555
13.5 Digital Filtering in the Time Domain 563
13.6 Linear Prediction and Linear Predictive Coding 569
13.7 Power Spectrum Estimation by the Maximum Entropy (All Poles) Method 577
13.8 Spectral Analysis of Unevenly Sampled Data 580
13.9 Computing Fourier Integrals Using the FFT 589
13.10 Wavelet Transforms 596
13.11 Numerical Use of the Sampling Theorem 611

14 Statistical Description of Data 614
14.0 Introduction 614
14.1 Moments of a Distribution: Mean, Variance, Skewness, and So Forth 615
14.2 Do Two Distributions Have the Same Means or Variances? 620
14.3 Are Two Distributions Different? 625
14.4 Contingency Table Analysis of Two Distributions 633
14.5 Linear Correlation 641
14.6 Nonparametric or Rank Correlation 644
14.7 Do Two-Dimensional Distributions Differ? 650
14.8 Savitzky-Golay Smoothing Filters 655

15 Modeling of Data 661
15.0 Introduction 661
15.1 Least Squares as a Maximum Likelihood Estimator 662
15.2 Fitting Data to a Straight Line 666
15.3 Straight-Line Data with Errors in Both Coordinates 671
15.4 General Linear Least Squares 676
15.5 Nonlinear Models 686
15.6 Confidence Limits on Estimated Model Parameters 694
15.7 Robust Estimation 704

16 Integration of Ordinary Differential Equations 712
16.0 Introduction 712
16.1 Runge-Kutta Method 715
16.2 Adaptive Stepsize Control for Runge-Kutta 719
16.3 Modified Midpoint Method 727
16.4 Richardson Extrapolation and the Bulirsch-Stoer Method 729
16.5 Second-Order Conservative Equations 737
16.6 Stiff Sets of Equations 739
16.7 Multistep, Multivalue, and Predictor-Corrector Methods 751

17 Two Point Boundary Value Problems 756
17.0 Introduction 756
17.1 The Shooting Method 760
17.2 Shooting to a Fitting Point 762
17.3 Relaxation Methods 765
17.4 A Worked Example: Spheroidal Harmonics 775
17.5 Automated Allocation of Mesh Points 785
17.6 Handling Internal Boundary Conditions or Singular Points 787

18 Integral Equations and Inverse Theory 790
18.0 Introduction 790
18.1 Fredholm Equations of the Second Kind 793
18.2 Volterra Equations 796
18.3 Integral Equations with Singular Kernels 799
18.4 Inverse Problems and the Use of A Priori Information 806
18.5 Linear Regularization Methods 811
18.6 Backus-Gilbert Method 818
18.7 Maximum Entropy Image Restoration 821

19 Partial Differential Equations 829
19.0 Introduction 829
19.1 Flux-Conservative Initial Value Problems 836
19.2 Diffusive Initial Value Problems 849
19.3 Initial Value Problems in Multidimensions 855
19.4 Fourier and Cyclic Reduction Methods for Boundary Value Problems 859
19.5 Relaxation Methods for Boundary Value Problems 865
19.6 Multigrid Methods for Boundary Value Problems 873

20 Less-Numerical Algorithms 891
20.0 Introduction 891
20.1 Diagnosing Machine Parameters 891
20.2 Gray Codes 896
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>20.3 Cyclic Redundancy and Other Checksums</td>
<td>898</td>
</tr>
<tr>
<td>20.4 Huffman Coding and Compression of Data</td>
<td>906</td>
</tr>
<tr>
<td>20.5 Arithmetic Coding</td>
<td>912</td>
</tr>
<tr>
<td>20.6 Arithmetic at Arbitrary Precision</td>
<td>916</td>
</tr>
<tr>
<td>References</td>
<td>927</td>
</tr>
<tr>
<td>Appendix A: Table of Function Declarations</td>
<td>931</td>
</tr>
<tr>
<td>Appendix B: Utility Routines and Classes</td>
<td>939</td>
</tr>
<tr>
<td>Appendix C: Converting to Single Precision</td>
<td>957</td>
</tr>
<tr>
<td>Index of Programs and Dependencies</td>
<td>959</td>
</tr>
<tr>
<td>General Index</td>
<td>972</td>
</tr>
</tbody>
</table>
Preface to the C++ Edition

C++ has gradually become the dominant language for computer programming, displacing C and Fortran even in many scientific and engineering applications. This version of Numerical Recipes contains the entire text of the Second Edition with all the programs presented in C++.

C++ poses special problems for numerical work. In particular, it is difficult to treat vectors and matrices in a manner that is simultaneously efficient and yet allows programming with high-level constructs. The fact that there is still no universally accepted standard library for doing this makes the problem even more difficult for authors of a book like this one. In Chapter 1 and the Appendices we describe how we have solved this problem. The default option is for you, the reader, to use a very simple class library that we provide. You can be up and running in a few minutes. We also show you how you can alternatively use any other matrix/vector class library of your choosing. This may take you a few minutes to set up the first time, but thereafter will provide transparent access to the Recipes with essentially no loss in efficiency.

We have taken this opportunity to respond to a clear consensus from our C readers, and converted all arrays and matrices to be “zero-based.” We have also taken this opportunity to fix errors in the text and programs that have been reported to us by our readers. There are too many people to acknowledge individually, but to all who have written to us we are very grateful.

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